Arvind Kr. Yadav
Asst. Prof., Deptt. of Maths
Raja Singh College Siwan

B. Sc. Ist
Differential Calculus
(Paper IInd)
Lecture - 5

Radius of curvature at the origin :

The following methods are used to find the radius of curvature at the origin :

(i) Method of substitution :- We have

$$\rho = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \left(\frac{d^2y}{dx^2}\right) \quad \text{--- (1)}$$

We find the $\left(\frac{dy}{dx}\right)_{(0,0)}$ and $\left(\frac{d^2y}{dx^2}\right)_{(0,0)}$ by putting $x=0$ and $y=0$ in their values. These values putting in equation (1); we obtain the value of radius of curvature at (0, 0).

(ii) Method of expansion :- Let $y = f(x)$ be the curve. Since it passes through the origin (0, 0); $\therefore y_{(0,0)} = f(0) = 0$.

By Maclaurin's series; we have

$$y = f(0) + x f'(0) + \frac{x^2}{\underline{2}} f''(0) + \text{———}$$

$$y = x f'(0) + \frac{x^2}{\underline{2}} f''(0) + \cdots \quad (\text{using } f(0)=0) \quad \text{——— (II)}$$

Now; if y can be expanded in ascending powers of x by trigonometrical or algebraic methods, then we have

$$y = px + \frac{qx^2}{\underline{2}} + \text{——— ——— (III)}$$

Comparing (II) & (III), we have $p = f'(0) = (y_1)_0$
and $q = f''(0) = (y_2)_0$

Using in (1), we have

$$\rho\,(\text{at the origin}) = \frac{(1+p^2)^{3/2}}{q}.$$

Example ①:- Show that the radius of curvature to the curve $y = 6x + 5x^2 + x^3$ at the origin is $\frac{37\sqrt{37}}{10}$.

Solution:- We have $y = 6x + 5x^2 + x^3$ ——— ①

Differentiating ①, w.r.to x; we have

$$\frac{dy}{dx} = 6 + 10x + 3x^2 \text{ ——— (ii)}$$

$$\left(\frac{dy}{dx}\right)_{(0,0)} = (y_1)_{(0,0)} = 6 + 10 \times 0 + 3 \times 0$$

$$(y_1)_0 = 6 \text{ ——— (iii)}$$

Differentiating (ii), w.r.t. x; we have

$$\frac{d^2y}{dx^2} = 0 + 10 \times 1 + 6 \times x \text{ ——— (iv)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(0,0)} = (y_2)_0 = 0 + 10 \times 1 + 6 \times 0 = 10 \text{ — (v)}$$

$$\rho \,(\text{at origin}) = \frac{\{1 + (y_1)_0^2\}^{3/2}}{(y_2)_0}$$

$$= \frac{\{1+6^2\}^{3/2}}{10} = \frac{(37)^{3/2}}{10} = \frac{(37)^{1+\frac{1}{2}}}{10}$$

$$\rho = \frac{37\sqrt{37}}{10}$$ (Hence proved).

Example ②:- Show that the radius of curvature of the curve $y^2 = x^2\left(\frac{a+x}{a-x}\right)$ at the origin are $\pm a\sqrt{2}$.

Solution:- We have $y^2 = x^2\left(\frac{a+x}{a-x}\right) \Rightarrow y = \pm x\sqrt{\frac{a+x}{a-x}}$

$$\Rightarrow y = \pm x(a+x)^{1/2}(a-x)^{-1/2}$$

$$= \pm x\left(1+\frac{x}{a}\right)^{1/2}\left(1-\frac{x}{a}\right)^{-1/2}$$

$$= \pm x\left\{1+\frac{1}{2}\cdot\frac{x}{a}-\frac{1}{8}\cdot\frac{x^2}{a^2}+\cdots\right\}\left\{1+\frac{1}{2}\cdot\frac{x}{a}+\frac{3}{8}\cdot\frac{x^2}{a^2}+\cdots\right\}$$

$$= \pm x\left(1+\frac{x}{a}+\cdots\right)$$

$$= \pm\left(x+\frac{x^2}{a}+\cdots\right)$$

which is of the form $y = px + \frac{qx^2}{2} + \text{—}$

comparing, we have

$$p = \pm 1 = (y_1)_0$$

$$\& \; q = \pm\frac{2}{a} = (y_2)_0$$

$$\therefore \rho(\text{at origin}) = \frac{(1+p^2)^{3/2}}{q}$$

$$= \frac{(1+1)^{3/2}}{\pm 2/a} = \pm a\sqrt{2}$$

Exercise :- Find the radius of curvature at origin for the curve

(i) $y = x^4 - 4x^3 - 18x^2$ (ii) $a(y^2 - x^2) = x^3$

(iii) $y^2 + 3xy + 2x^2 + x^3 + y^4 = 0$.